Goals

- to introduce CS modules relative to a torsion theory
  - $\tau$-CS modules: a generalization of CS modules,
  - s-$\tau$-CS modules: a specialization of CS modules,
- to investigate their relationship with $\tau$-injective, $\tau$-simple and $\tau$-uniform modules,
- to compare them with alternative generalizations (e.g. $\tau$-complemented ($\tau$-injective) modules),
- to decompose (s-)\$\tau$-CS modules into indecomposables,
- when is a direct sum of (s-)\$\tau$-CS modules (s-)\$\tau$-CS?
Preliminary Concepts: Notation

- **modules**: left $R$-modules for some unitary ring $R$.

- **homomorphisms**: $R$-module homomorphisms.

- **$R$-Mod**: class of all left $R$-modules.

- **$\tau = (T, F)$**: hereditary torsion theory on $R$-Mod.
  - $T$: class of all $\tau$-torsion modules.
  - $F$: class of all $\tau$-torsionfree modules.

- **$t_{\tau}(M)$**: the $\tau$-torsion submodule of a module $M$.

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Preliminary Concepts

**Def.** A submodule $N$ of $M$ is called $\tau$-dense ($\tau$-pure)
- if $M/N$ is $\tau$-torsion, (if $M/N$ is $\tau$-torsionfree),
- we then write $N \leq_{\tau-d} M$, (we then write $N \leq_{\tau-p} M$),
- $D_{\tau}(M) = \{\tau$-dense sub.}, ($P_{\tau}(M) = \{\tau$-pure sub.}$)$

**Def.** The $\tau$-pure closure of $N$ in $M$ is:
- $\text{Cl}_{\tau}^M(N) = N^c = \bigcap\{K \in P_{\tau}(M) \mid N \leq K\}$

**Def.** A module $M$ is called $\tau$-presimple if it has
- exactly two $\tau$-pure submodules, i.e., $t_{\tau}(M)$ and $M$.

**Def.** A module $M$ is called $\tau$-simple if:
- it is $\tau$-presimple and $\tau$-torsionfree.
Preliminary Concepts

**Prop.** A module $M$ is $\tau$-simple if and only if
- $M$ is nonzero $\tau$-torsionfree, and
- every nonzero submodule of $M$ is $\tau$-dense in $M$.

**Def.** A module $M$ is called $\tau$-compact if:
- every nonzero submodule of $M$ is $\tau$-dense in $M$.

**Prop.** A module $M$ is $\tau$-compact if and only if
- $M$ is $\tau$-torsion or $\tau$-simple.

**Def.** A submodule $N$ of $M$ is called $\tau$-essential if:
- $N$ is essential and $\tau$-dense in $M$.

$\tau$-uniform and $\tau$-complement modules

**Def.** A module $M$ is called $\tau$-uniform if:
- every nonzero submodule of $M$ is $\tau$-essential in $M$.

$\Leftrightarrow$ $M$ is uniform and $\tau$-compact.

$\Leftrightarrow$ Either $M$ is $\tau$-simple or both uniform and $\tau$-torsion.

Let $K$ and $L$ be submodules of a module $M$. Then:

**Def.** $K$ is called a $\tau$-complement of $N$ in $M$ if
- $K$ is maximal with respect to $\leq$ in the class
  \[
  \{ L \leq M \mid L \cap N = 0 \text{ and } L \oplus N \in \mathcal{D}_\tau(M) \}
  \]

**Def.** $K$ is called a $\tau$-complement submodule of $M$ if
- $K$ is a $\tau$-complement of some submodule $N$ of $M$
Existence of $\tau$-complements

**Prop.** A submodule $N$ of $M$ has a $\tau$-complement in $M$ if
$\Leftrightarrow (\exists K \leq M)$ such that $K \cap N = 0$ and $K \oplus N \in D_\tau(M)$

**Def.** A torsion theory $\tau$ is called **cohereditary** if
- $\mathcal{F}$ is closed under homomorphic images

**Note:** $\tau$-complements do NOT always exist.
- If $N \leq_e M$ and $N \not\in D_\tau(M)$ then $N$ has no $\tau$-complement in $M$.
- Let $\tau = (\mathcal{T}, \mathcal{F})$ be a **cohereditary** torsion theory.
  - If $M \in \mathcal{F}$ then $(\forall M \neq N \leq_e M)$ $N$ has no $\tau$-complement in $M$.

$\tau$-M-Injective Modules

**Def.** A module $E$ is **$\tau$-M-injective**, for a module $M$, if:
- every hom. from a $\tau$-dense submodule $N$ of $M$ to $E$ extends to a homomorphism from $M$ to $E$.
- *i.e.* for any $N \in D_\tau(M)$ and any $f \in \text{Hom}_R(N, E)$ there is a $g \in \text{Hom}_R(M, E)$ such that:

$$
\begin{array}{ccc}
0 & \longrightarrow & N & \xrightarrow{i} & M \\
\downarrow f & & \downarrow & & \downarrow (\exists g)
g|_N = f \\
E & & \longleftarrow & & \end{array}
$$
More relatively injective modules

**Def.** A module $E$ is **τ-quasi-injective**, if:
- $E$ is $τ$-$E$-injective.

**Def.** A module $E$ is **τ-injective**, if:
- $E$ is $τ$-$M$-injective for any module $M$.

**Def.** A family $\{E_i | i \in I\}$ is relatively **τ-injective**, if:
- $E_i$ is $τ$-$E_j$-injective for any $i \neq j \in I$.

**Def.** $E$ is an **τ-injective hull** of $M$, denoted by $E_τ(M)$
- $E$ is a minimal $τ$-injective extension of $M$, $⇔$
- $E$ is a maximal $τ$-essential extension of $M$, $⇔$
- $E$ is a $τ$-injective $τ$-essential extension of $M$.

τ-CS modules

**Def.** A module $M$ is called **τ-CS** if any one of the following equivalent conditions holds:

(1) Every $τ$-dense submodule of $M$ is essential in a direct summand of $M$.

(2) Every $τ$-dense submodule of $M$ is $τ$-essential in a direct summand of $M$.

(3) Every $τ$-dense, $τ$-essentially closed submodule of $M$ is a direct summand of $M$.

(4) Every $τ$-dense, essentially closed submodule of $M$ is a direct summand of $M$.

**Prop.** By (1) or (4) we get: Every CS module is $τ$-CS.
**Quasi-continuous modules**

A module \( M \) is called *quasi-continuous* if it satisfies:

1. **(C1)** \((\forall N \leq M) \ (\exists N^* \leq M)\) such that \( N \leq_e N^* \leq_\oplus M\).
2. **(C3)** For any \( K, L \leq M \), if \( K \leq_\oplus M \), \( L \leq_\oplus M \) and \( K \cap L = 0 \) then \( K \oplus L \leq_\oplus M\).

**Prop.** The following are equivalent for a module \( M \):

- \( M \) is quasi-continuous.
- If \( C \) and \( D \) are complements of each other in \( M \) then \( C \oplus D = M \).
- For any \( f^2 = f \in \text{End}_R(E(M)) \) we have \( f(M) \leq M \).
- If \( E(M) = \bigoplus_{i \in I} E_i \) then \( M = \bigoplus_{i \in I} (M \cap E_i) \).

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**τ-Quasi-continuous modules**

**Def.** A module \( M \) is said to be *τ-quasi-continuous* if

- For any \( f^2 = f \in \text{End}_R(E_\tau(M)) \) we have \( f(M) \leq M \).

**Prop.** Let \( M \) be a module. We have \((1) \Rightarrow (2) \Rightarrow (3)\).

1. \( M \) is \( \tau \)-quasi-continuous.
2. If \( E_\tau(M) = \bigoplus_{i \in I} E_i \) then \( M = \bigoplus_{i \in I} (M \cap E_i) \).
3. **(τ-C1)** for any \( N \in D_\tau(M) \) there exists an \( N^* \leq M \) such that \( N \leq_\tau-e N^* \leq_\oplus M \).

**τ-C3** for any \( K_1 \leq_\oplus M \), \( K_2 \leq_\oplus M \), if \( K_1 \cap K_2 = 0 \) and \( K_1 \oplus K_2 \in D_\tau(M) \), then \( K_1 \oplus K_2 \leq_\oplus M \)
Examples of $\tau$-quasi-continuous modules

**Prop.** A module $M$ is $\tau$-quasi-injective if and only if

- For any $f \in \text{End}_R(E_\tau(M))$ we have $f(M) \leq M$. Thus:

$\{\tau\text{-injective}\} \subseteq \{\tau\text{-quasi-injective}\} \subseteq \{\tau\text{-quasi-continuous}\}$

**Prop.** Since (C1) $\Rightarrow$ (C1) and (C3) $\Rightarrow$ (C3) we have:

- $\{\text{quasi-continuous}\} \subseteq \{\tau\text{-quasi-continuous}\} \subseteq \{\tau\text{-CS}\}$

**Def.** A module $M$ is called $s\tau$-CS (strongly $\tau$-CS) if any one of the following equivalent conditions holds:

1. Every submodule of $M$ is $\tau$-essential in a direct summand of $M$.
2. Every $\tau$-essentially closed submodule of $M$ is a direct summand of $M$.

Examples of $s\tau$-CS modules

**Prop.** The following are equivalent for a module $M$:

1. $M$ is $\tau$-uniform.
2. $M$ is $s\tau$-CS and indecomposable.
3. $M$ is CS, indecomposable and $\tau$-compact.
4. $M$ is $\tau$-CS, indecomposable and $\tau$-compact.

Examples of $s\tau$-CS modules:

- $\{\tau\text{-simple}\} \subseteq \{\tau\text{-uniform}\} \subseteq \{s\tau\text{-CS}\}$
- $\{\tau\text{-torsion }\tau\text{-CS}\} \subseteq \{\tau\text{-compact }\tau\text{-CS}\} \subseteq \{s\tau\text{-CS}\}$
Decomposition of $\tau$-CS modules

**Okado:** Let $M$ be CS with the ACC on $\{ (0 : x) \mid x \in M \}$.

- Then $M = \bigoplus_i U_i$, with the $U_i$ indecomposable uniform.

**Note:** A refinement of Masaike and Horigome, $[(1) \iff (3)]$

**Theorem:** The following statements are equivalent:

1. $R$ has ACC on $\tau$-dense left ideals, i.e., $D_\tau(R)$ has ACC.
2. Each $\tau$-torsion $\tau$-CS $R$-module is a direct sum of $\tau$-uniform ($\tau$-CS) submodules.
3. Each $\tau$-torsion $\tau$-injective $R$-module is a direct sum of $\tau$-uniform ($\tau$-injective) submodules.

Direct Sum of s-$\tau$-CS modules

**Lem.** Let $M = M_1 \oplus M_2$, with $M_1$, $M_2$ $\tau$-compact s-$\tau$-CS.

Then the following statements are equivalent:

1. $M$ is s-$\tau$-CS.
2. For any $\tau$-essentially closed submodule $N$ of $M$, if $N \cap M_1 = 0$ or $N \cap M_2 = 0$ then $N \leq \bigoplus M$.

**Thm.** Let $M = M_1 \oplus \cdots \oplus M_n$ be a finite direct sum of relatively $\tau$-injective $\tau$-compact modules.

- Then $M$ is s-$\tau$-CS if and only if each $M_i$ is s-$\tau$-CS.

**Cor.** Since $\{ \tau$-simple $\} \subseteq \{ \tau$-uniform $\} \subseteq \{ \tau$-compact $\}$

- Every finite direct sum of relatively $\tau$-injective, $\tau$-simple (or $\tau$-uniform) modules is s-$\tau$-CS.
**τ-Okado & τ-complemented modules**

**Thm.** [τ-Okado] The following statements are equivalent:

1. $R$ is $\tau$-Noetherian, i.e., $R$ has ACC on $\tau$-pure left ideals.

2. Every $\tau$-torsionfree CS $R$-module is a direct sum of indecomposable submodules.

3. Every $\tau$-torsionfree injective $R$-module is a direct sum of indecomposable submodules.

**Def.** [SVV] A module $M$ is called $\tau$-complemented if

- $(\forall N \leq M)(\exists N^* \leq M)$ such that $N \leq_{\tau-d} N^* \leq \bigoplus M$.

**Note:** If $M \in \mathcal{F}$ and $\tau$-complemented then $M$ is CS.

**Example:** Every $s-\tau$-CS module is $\tau$-complemented.

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**Okado: $\tau$-complemented versus s-$\tau$-CS**

**Thm.** [SVV] For a $\tau$-complemented $R$-module $M$, (1) $\iff$ (2)

1. $M = T \oplus \bigoplus_i S_i$ with $T \in T$, $(\forall i \in I) S_i$ is $\tau$-simple

2. $R$ has ACC on $\{(0 : x) \mid x \in M/t_\tau(M)\}$

**Lemma** For a $s-\tau$-CS $R$-module $M$:

- If $R$ has ACC on $\{(0 : x) \mid x \in M/t_\tau(M)\}$ and
  - on $\{(0 : x) \mid x \in t_\tau(M)\}$
  
  then $M = \bigoplus_i U_i$ s.t. $(\forall i \in I) U_i$ is $\tau$-uniform ($s-\tau$-CS)

**Corollary**

- If $R$ has ACC on $P_\tau(R)$ ($\tau$-pure ideals) and
  - on $D_\tau(R)$ ($\tau$-dense ideals)
  
  then every s-$\tau$-CS $R$-module $= \bigoplus \tau$-uniform submodules
**Chain conditions and s-τ-CS modules**

**Prop.** For a s-τ-CS $R$-module $M$:
- If $M$ is a direct sum of $τ$-uniform (s-τ-CS) submodules then $R$ has ACC on \( \{ (0 : x) \mid x \in M/t_τ(M) \} \)

**Note:** If in addition $t_τ(M)$ is non-singular then
- $R$ has ACC on \( \{ (0 : x) \mid x \in t_τ(M) \} \)

**Prop.** If every s-τ-CS $R$-module $= \bigoplus τ$-uniform (s-τ-CS)
- then $R$ has ACC on $D_τ(R) = \{ τ$-dense ideals\}

**Note:** It is an open question whether in addition
- $R$ has ACC on $P_τ(R) = \{ τ$-pure ideals\}

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**s-τ-CS and τ-complemented τ-injective modules**

**Def.** [Crivei] A module $M$ is called minimal $τ$-injective if
- For any $0 \neq N \subseteq M$ we have $M = E_τ(N)$

or $M$ is minimal $τ$-injective $⇔ M$ is $τ$-uniform $τ$-injective

**Def.** [MH] We call a module $M$ $τ$-completely decomposable
- if $M = \bigoplus_I M_i$ where each $M_i$ is minimal $τ$-injective

**Prop.** If $M$ is $τ$-complemented $τ$-injective then $M$ is s-τ-CS

$M$ is $τ$-complemented $τ$-injective $⇔ M$ is s-τ-CS $τ$-injective

**Thm.** [Crivei] If $R$ has ACC on $D_τ(R)$ and $P_τ(R)$ then
- an $R$-module $M$ is $τ$-completely decomposable
  if and only if $M$ is $τ$-complemented $τ$-injective
Some open questions

Open question 1
• If $R$ has ACC on $\mathcal{P}_\tau(R)$ and $\mathcal{D}_\tau(R)$ then
  is every direct sum of $\tau$-uniform (s-$\tau$-CS) modules s-$\tau$-CS?

Note: If the answer to the above is positive then
• we have a refinement of the previous theorem by Crivei
  i.e. we have a positive answer to the following question:

Open question 2
• If $R$ has ACC on $\mathcal{P}_\tau(R)$ and $\mathcal{D}_\tau(R)$ then
  an $R$-module $M$ is $\tau$-completely decomposable
  if and only if $M$ is s-$\tau$-CS